Project systems theory

Final exam 2016–2017, Thursday 26 January 2017, 9:00 – 12:00

Problem 1

(4 + 8 + 3 = 15 points)

A model for population dynamics is given by

$$\begin{aligned} \dot{p}(t) &= \Big(2 - \tfrac{1}{6}p(t) - \tfrac{1}{4}q(t)\Big)p(t), \\ \dot{q}(t) &= \Big(\tfrac{1}{4}p(t) - \tfrac{1}{2} - u(t)\Big)q(t), \\ y(t) &= p(t) + q(t), \end{aligned}$$

with p the number of prey and q the number of predators. The input u denotes the fraction of predators that is killed by hunting, whereas y measures the total population.

(a) Let $u(t) = 1$ for all $t \in \mathbb{R}$. Determine the equilibrium solution $p(t) = \bar{p}$, $q(t) = \bar{q}$ for which $\bar{p} > 0$ and $\bar{q} > 0$.

(b) Linearize the system around the equilibrium solution obtained in (a).

(c) Is the linearized system (internally) stable?

Problem 2

(15 points)

Consider the polynomial

$$p(\lambda) = \lambda^4 + a\lambda^3 + 4\lambda^2 + 2a\lambda + b,$$

with a and b real numbers. Determine all values of a and b such that the polynomial is stable.

Problem 3

(4 + 10 + 6 = 20 points)

Consider the system

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u.$$

(a) Is the system controllable?

(b) Find a nonsingular matrix T and real numbers α_1, α_2, α_3 such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(c) Use the matrix T from problem (b) to obtain a state feedback of the form $u = Fx$ such that the closed-loop system matrix $A + BF$ has eigenvalues at -1 and -2.

Problem 4 (3 + 3 + 4 + 4 + 3 + 3 = 20 points)

Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ -3 & 1 & 4 \\ -1 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x.$$

Answer the following questions and explain your answers:

(a) Is the system (internally) stable?

(b) Is the system controllable?

(c) Determine the reachable subspace.

(d) Is the system stabilizable?

(e) Is the system observable?

(f) Is the system detectable?

Problem 5 (20 points)

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. Let $\mathcal{W}$ and $\mathcal{N}$ be the reachable subspace and unobservable subspace, respectively, and denote the impulse response as

$$h(t) = Ce^{At}B.$$

Show that $h(t) = 0$ for all $t \in \mathbb{R}$ if and only if $\mathcal{W} \subset \mathcal{N}$.

(10 points free)